SOLUTION OF A DIRECT PROBLEM OF HEAT EXCHANGE BETWEEN A COOLED TURBINE BLADE AND A GAS BY THE PIVOT METHOD

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Using the pivot method we solve a third-order differential equation that describes the heat exchange of a gas with a cooled turbine blade.

The temperature variation along the length of a cooled turbine blade is described by a third-order differential equation with variable coefficients [1, 2]. Variants of the analytical solution of the indicated equation are developed in [1, 2]. It is also possible to use difference methods [3], which, in principle, are more universal. The method proposed in [3] has a shortcoming related to the difficulty of accurately determining the initial data for calculations at the blade root satisfying the boundary condition at the vertex [4].

1. Differential Equation for Determining the Temperature of the Cooled Blade

Under thermal conditions the temperature distribution along the length of the cooled blade is determined from the following relations:

a) the heat balance in element dx

$$\left(\lambda F\frac{dT_b}{dx}\right)_{x+dx}+\alpha_g u_g(T_g^*-T_b)\,dx=\alpha_a u_a(T_b-T_a^*)\,dx+\left(\lambda F\frac{dT_b}{dx}\right)_x; \quad (1)$$

b) the equations of heating of the cooling air flowing by element dx, taking account of the effect of the centrifugal forces

$$G_a' c_{pa} dT_a^* = \alpha_a u_a (T_b - T_a^*)\,dx + G_a' c_{pa} dT_{au}. \quad (2)$$

From (1) and (2) we have [1]

$$\frac{d^2}{dx^2}\left(\frac{\lambda F}{\lambda_r F_r}\cdot\frac{d\Theta}{dx}\right)+\frac{\alpha_a u_a l}{G_a' c_{pa}}\cdot\frac{d}{dx}\left(\frac{\lambda F}{\lambda_r F_r}\cdot\frac{d\Theta}{dx}\right)-\left(\frac{\alpha_{g\,av}u_{g\,av}l^2}{\lambda_r F_r}\cdot\frac{\alpha_g u_g}{\alpha_{r\,av}u_{r\,av}}+\frac{\alpha_{a\,av}u_{a\,av}l^2}{\lambda_r F_r}\right)\frac{d\Theta}{dx}$$
$$-\left[\frac{\alpha_{g\,av}u_{g\,av}l^2}{\lambda_r F_r}\cdot\frac{\alpha_a u_a l}{G_r' c_{pa}}\cdot\frac{\alpha_g u_g}{\alpha_{g\,av}u_{g\,av}}+\frac{\alpha_{g\,av}u_{g\,av}l^2}{\lambda_r F_r}\cdot\frac{d}{dx}\left(\frac{\alpha_g u_g}{\alpha_{g\,av}u_{g\,av}}\right)\right]\Theta$$
$$=-\frac{\alpha_{g\,av}u_{g\,av}l^2}{\lambda_r F_r}\cdot\frac{\alpha_g u_g}{\alpha_{g\,av}u_{g\,av}}\cdot\frac{d\Theta_g^*}{dx}-\left[\frac{\alpha_{g\,av}u_{g\,av}l^2}{\lambda_r F_r}\cdot\frac{\alpha_a u_a l}{G_a' c_{pa}}\right.$$
$$\left.\times\frac{\alpha_g u_g}{\alpha_{g\,av}u_{g\,av}}+\frac{\alpha_{g\,av}u_{g\,av}l^2}{\lambda_r F_r}\cdot\frac{d}{dx}\left(\frac{\alpha_g u_g}{\alpha_{g\,av}u_{g\,av}}\right)\right]\Theta_r^*-\frac{\alpha_{a\,av}u_{a\,av}l^2}{\lambda_r F_r}\cdot\frac{\omega^2 l^2}{c_{pa}T_{g\,av}^*}\left(\frac{r_r}{l}+\bar{x}\right). \quad (3)$$

From (3) we can obtain a differential equation of the form

$$\Theta'''K_3^*+\Theta''K_2^*+\Theta'K_1^*+\Theta K_0^*=K^*, \quad (4)$$

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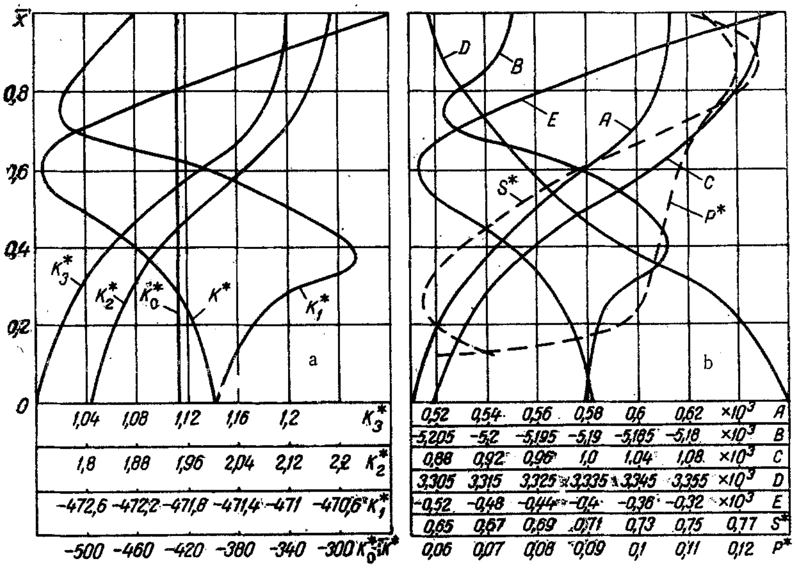

Fig. 1. Coefficients of: a) the differential equations; b) the recurrence relations.

where

$$K_3^* = \frac{\lambda F}{\lambda_r F_r}; \quad K_2^* = \frac{\alpha_a u_a l}{G_a' c_{pa}};$$

$$K_1^* = \frac{d^2}{d\bar{x}^2}\left(\frac{\lambda F}{\lambda_r F_r}\right) + \frac{\alpha_a u_a l}{G_a' c_{pa}} \cdot \frac{d}{d\bar{x}}\left(\frac{\lambda F}{\lambda_r F_r}\right) - \frac{\alpha_{g\,av} u_{g\,av} l^2}{\lambda_r F_r} \cdot \frac{\alpha_g u_g}{\alpha_{g\,av} u_{g\,av}} - \frac{\alpha_{a\,av} u_{a\,av} l^2}{\lambda_r F_r};$$

$$K_0^* = -\frac{\alpha_{g\,av} u_{g\,av} l^2}{\lambda_r F_r} \cdot \frac{\alpha_a u_a l}{G_a' c_{pa}} \cdot \frac{\alpha_g u_g}{\alpha_{g\,av} u_{g\,av}} - \frac{\alpha_{g\,av} u_{g\,av} l^2}{\lambda_r F_r} \cdot \frac{d}{d\bar{x}}\left(\frac{\alpha_g u_g}{\alpha_{g\,av} u_{g\,av}}\right);$$

$$K^* = -\frac{\alpha_{g\,av} u_{g\,av} l^2}{\lambda_r F_r} \cdot \frac{\alpha_g u_g}{\alpha_{g\,av} u_{g\,av}} \cdot \frac{d\Theta_g^*}{d\bar{x}} - \Theta_g^*\left[\frac{\alpha_{g\,av} u_{g\,av} l^2}{\lambda_r F_r} \cdot \frac{\alpha_a u_a l}{G_a' c_{pa}}\right.$$

$$\left.\times \frac{\alpha_g u_g}{\alpha_{g\,av} u_{g\,av}} + \frac{\alpha_{g\,av} u_{g\,av} l^2}{\lambda_r F_r} \cdot \frac{d}{d\bar{x}}\left(\frac{\alpha_g u_g}{\alpha_{g\,av} u_{g\,av}}\right)\right] + \frac{\alpha_{a\,av} u_{a\,av} l^2}{\lambda_r F_r} \cdot \frac{\omega^2 l^2}{c_{pa} T^*_{g\,av}}\left(\frac{r_r}{l} + \bar{x}\right).$$

The coefficients K_3^*, K_2^*, K_1^*, K_0^*, and K^* have a complicated character of variation; in the general case it is not possible to solve the sought differential equation analytically. Results of a numerical calculation of the variation of the coefficients of the differential equation for a specific blade are given in Fig. 1a. To calculate the coefficients we took the following values: u_g = 116 mm; u_a = 83 mm; F = $2.07 \cdot 10^{-4}$ m²; r_r = 262.8 mm; ω = 1256 1/sec; G_a' = 0.00622 kg/sec; c_{pa} = 1005 J/kg · deg; l = 92 mm; τ = 1000 h; h = 0.125; α_g = 1060 W/m² · deg; we used the material ÉI-612; $T^*_{g\,av}$ = 1145°K; $T^*_{a\,r}$ = 510°K; $\alpha_a u_a/\alpha_g u_g = 1$; $\lambda = \lambda(\bar{x})$ was commonly in conformity with the characteristics of the material and the values of the temperature T_b based on preliminary calculations; $\Theta_g^* = \Theta_g^*(\bar{x})$ (see Fig. 2 below).

2. Solution of Equation

We replace the derivatives by the finite-difference relations:

$$(\Theta')_n \approx \frac{1}{h}(\Theta_{n+1} - \Theta_{n-1}); \tag{5}$$

$$(\Theta'')_n \approx \frac{1}{h^2}(\Theta_{n+1} - 2\Theta_n + \Theta_{n-1}); \tag{6}$$

$$(\Theta''')_n \approx \frac{1}{h^3}(\Theta_{n+2} - 3\Theta_{n+1} + 3\Theta_n - \Theta_{n-1}). \tag{7}$$

From (4)-(7) for the cross section 0 we obtain

$$A_0\Theta_2 + B_0\Theta_1 + C_0\Theta_0 + D_0\Theta_{-1} = E_0, \tag{8}$$

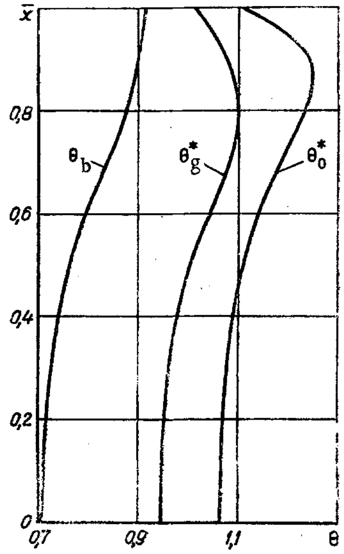

Fig. 2. Variation of the relative temperatures along the blade length.

where

$$A_0 = \frac{K_3^*}{h^3};\quad B_0 = \frac{K_1^*}{h} + \frac{K_2^*}{h^2} - 3\frac{K_3^*}{h^3};\quad C_0 = K_0^* - 2\frac{K_2^*}{h^2} + 3\frac{K_3^*}{h^3};$$

$$D_0 = -\frac{K_1^*}{h} + \frac{K_2^*}{h^2} - \frac{K_3^*}{h^3};\quad E_0 = K^*.$$

In general form, for the n-th cross section, we have

$$A_n\Theta_{n+2} + B_n\Theta_{n+1} + C_n\Theta_n + D_n\Theta_{n-1} = E_n. \tag{9}$$

In the root cross section we are given the boundary conditions:

$$1.\ \Theta_r = \Theta_0;$$

$$2.\ \frac{1}{h^2}(\Theta_1 - 2\Theta_0 + \Theta_{-1}) = m_0 = \left(\frac{d^2\Theta}{d\bar{x}^2}\right)_0$$

(they are determined by means of preliminary calculations from (1) for given Θ_{a0}^*; $\lambda = \lambda(\bar{x})$; $F = F(\bar{x})$). From the second boundary condition

$$\bar{B}_0\Theta_1 + \bar{C}_0\Theta_0 + \bar{D}_0\Theta_{-1} = \bar{E}_0. \tag{10}$$

Eliminating Θ_{-1} from (8) and (10), we obtain

$$\bar{D}_0A_0\Theta_2 + (B_0\bar{D}_0 - \bar{B}_0D_0)\Theta_1 + (C_0\bar{D}_0 - \bar{C}_0D_0)\Theta_0 = E_0\bar{D}_0 - \bar{E}_0D_0. \tag{11}$$

From (11) we determine $\Theta_1 = \Theta_1(\Theta_2)$ in the form

$$\Theta_1 = S_1^* + P_1^*\Theta_2, \tag{12}$$

where

$$S_1^* = \frac{E_0\bar{D}_0 - \bar{E}_0D_0}{B_0\bar{D}_0 - \bar{B}_0D_0} - \frac{C_0\bar{D}_0 - \bar{C}_0D_0}{B_0\bar{D}_0 - \bar{B}_0D_0}\Theta_0; \tag{13}$$

$$P_1^* = -\frac{A_0\bar{D}_0}{B_0\bar{D}_0 - \bar{B}_0D_0}. \tag{14}$$

We write Eq. (9) for the cross section 1:

$$A_1\Theta_3 + B_1\Theta_2 + C_1\Theta_1 + D_1\Theta_0 = E_1. \tag{15}$$

Replacing Θ_1 in (15) and determining $\Theta_2 = \Theta_2(\Theta_3)$, from (12) we have

$$\Theta_2 = \frac{E_1 - D_1\Theta_0 - C_1S_1^*}{B_1 + C_1P_1^*} - \frac{A_1}{B_1 + C_1P_1^*}\Theta_3,$$

or

$$\Theta_2 = S_2^* + P_2^*\Theta_3, \tag{16}$$

where

$$S_2^* = \frac{E_1 - D_1\Theta_0 - C_1S_1^*}{B_1 + C_1P_1^*}; \tag{17}$$

$$P_2^* = -\frac{A_1}{B_1 + C_1P_1^*}. \tag{18}$$

We write Eq. (9) for the cross section 2 and we determine similarly $\Theta_3 = \Theta_3(\Theta_4)$, using (12) and (16):

$$\Theta_3 = \frac{E_2 - C_2S_2^* - D_2S_1^* - D_2P_1^*S_2^*}{B_2 + C_2P_2^* + D_2P_2^*P_1^*} - \frac{A_2}{B_2 + C_2P_2^* + D_2P_2^*P_1^*}\Theta_4,$$

or

$$\Theta_3 = S_3^* + P_3^*\Theta_4. \tag{19}$$

We write Eq. (9) for the cross section 3 and we determine $\Theta_4 = \Theta_4(\Theta_5)$, using (16) and (19):

$$\Theta_4 = \frac{E_3 - C_3S_3^* - D_3S_2^* - D_3P_2^*S_3^*}{B_3 + C_3P_3^* + D_3P_3^*P_2^*} - \frac{A_3}{B_3 + C_3P_3^* + D_3P_3^*P_2^*}\Theta_5, \tag{20}$$

or

$$\Theta_4 = S_4^* + P_4^*\Theta_5.$$

As a result of the calculations performed, we can develop a general relation for the determination of $\Theta_n = \Theta_n(\Theta_{n+1})$ in the form of the following recurrence relations:

$$\Theta_n = S_n^* + P_n^*\Theta_{n+1}; \quad (21)$$

$$\Theta_{n-1} = S_{n-1}^* + P_{n-1}^*\Theta_n, \quad (22)$$

where

$$S_n^* = \frac{E_{n-1} - S_{n-1}^*(C_{n-1} + D_{n-1}P_{n-2}^*) - S_{n-2}^*D_{n-1}}{B_{n-1} + C_{n-1}P_{n-1}^* + D_{n-1}P_{n-1}^*P_{n-2}^*};$$

$$P_n^* = -\frac{A_{n-1}}{B_{n-1} + C_{n-1}P_{n-1}^* + D_{n-1}P_{n-1}^*P_{n-2}^*}.$$

The boundary condition at the vertex of the blade $\Theta'_{\bar{x}=1}$ (we neglect the discharge of heat into the gas through the upper face of the blade).

We write Eq. (9) for the cross section (n−1):

$$A_{n-1}\Theta_{n+1} + B_{n-1}\Theta_n + C_{n-1}\Theta_{n-1} + D_{n-1}\Theta_{n-2} = E_{n-1}. \quad (23)$$

From the boundary condition for $\bar{x} = 1$ we obtain

$$\bar{A}_{nb}\Theta_{n+1} + \bar{C}_{nb}\Theta_{n-1} = \bar{E}_{nb}, \quad (24)$$

where

$$\bar{A}_{nb} = \frac{1}{2\bar{h}}; \quad \bar{C}_{nb} = -\frac{1}{2\bar{h}}; \quad \bar{E}_{nb} = 0.$$

From (23) and (24), eliminating Θ_{n+1} and taking into account that $\Theta_{n-2} = S_{n-2}^* + P_{n-2}^*\Theta_{n-1}$, we have

$$\Theta_{n-1} = \frac{\bar{A}_{nb}(E_{n-1} - D_{n-1}S_{n-2}^*) - \bar{E}_{nb}A_{n-1}}{\bar{A}_{nb}(C_{n-1} + D_{n-1}P_{n-2}^*) - A_{n-1}\bar{C}_{nb}} - \frac{\bar{A}_{nb}B_{n-1}}{\bar{A}_{nb}(C_{n-1} + D_{n-1}P_{n-2}^*) - A_{n-1}\bar{C}_{nb}}\Theta_n, \quad (25)$$

or

$$\Theta_{n-1} = S_{n-1}^{**} + P_{n-1}^{**}\Theta_n. \quad (26)$$

From (22) and (26) we determine Θ_n:

$$\Theta_n = \frac{S_{n-1}^* - S_{n-1}^{**}}{P_{n-1}^{**} - P_{n-1}^*}. \quad (27)$$

Knowing Θ_n, from Eq. (22) we can determine successively $\Theta_{n-1} \ldots \Theta_1$.

Results of a numerical calculation for the coefficients A, B, C, D, E, S*, and P* for a specific blade are shown in Fig. 1b (see above for the initial data).

Results of a numerical calculation of the temperature are given in Fig. 2. An increase in the number of sections did not lead to a noticeable refinement in the $\Theta_b = \Theta_b(\bar{x})$ dependence.

NOTATION

λ	is the coefficient of thermal conductivity of the material;
F	is the cross sectional area of the blade;
α_g, α_a	are the coefficients of heat exchange on the gas and air sides;
u_g, u_a	are the exterior and interior perimeters of the profile of the blade;
T_g^*, T_a^*	are the temperatures in the boundary layer on the gas and air sides;
$G_a^I = G_a/z$	is the air flow-rate through one blade (z is the number of blades);
c_{pa}	is the specific heat of air;
$dT_{au} = \omega^2(r_r + x)dx/c_{pa}$	is the heating of the air due to the compression by centrifugal forces at the section dx;
$\Theta = T/T_{gav}^*$	is the relative temperature.

Subscripts

r denotes the root of the blade;

av denotes the average diameter.

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